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Research Institute for Advanced Studies (RIAS)
7212 Bellona Avenue
Baltimore 12, Maryland

Fundamental Inequalities for Discrete and

Discontinuous Functional Equations

By

G. Stephen Jones*

Gronwall's lemma embodies an inequality which is often referred to, and justifiably so, as the fundamental inequality of differential equations. Its usefulness is felt from the very beginning when one encounters the most elementary theorem on the existence and uniqueness of solutions and their dependence on parameters and initial values. Furthermore, it continues to be an extremely simple yet effective tool as one proceeds to the more delicate and sophisticated considerations in the theory of perturbations and stability. A statement of this lemma usually proceeds somewhat as follows:

Lemma 1 (Gronwall's Lemma). Let x, f and z be real-valued piecewise continuous functions defined on a real interval $[a, b]$, and let z be nonnegative on this interval. If for all t in $[a, b]$

$$x(t) \leq f(t) + \int_a^t z(\tau)x(\tau)d\tau, \qquad (1)$$

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then

$$x(t) \leqq f(t) + \int_a^t z(\tau)f(\tau)\exp(\int_\tau^t z(s)ds)d\tau, \qquad (2)$$

for all t in $[a, b]$.

The proof follows in a completely straight-forward manner. We have only to observe that if we let $F(t)$ denote the integral in (1), then (1) implies $F'(t) - z(t)F(t) \leqq z(t)f(t)$ at all points of continuity of $z(t)x(t)$. Multiplying through by $\exp(-\int_a^t z(\tau)d\tau)$ and integrating from a to t it follows immediately that

$$\int_a^t z(\tau)f(\tau)\exp(\int_\tau^t z(s)ds)d\tau \geqq \int_a^t z(\tau)x(\tau)d\tau,$$

which, of course, implies (2). The advantage of inequality (2) over inequality (1) becomes apparent if we consider f and z as known functions and x as unknown. That is, inequality (2) gives us a completely known function which majorizes x. One may also observe that inequality (2) is the best possible resulting from inequality (1) in the sense that if inequality is replaced by equality in (1) then the same may be done in (2). Many applications of this lemma may be found in reference [2], [5], and [6] as well as in numerous other books and papers.

In this paper we shall present a series of lemmas which contain inequalities which may be used in the theory of finite difference equations and more general discontinuous functional equations in essentially the same capacity as the inequalities of Lemma 1 are used in the theory of differential equations. To illustrate the application of these lemmas several results concerning boundedness and stability of solutions are presented.

As one might expect, it is possible to construct lemmas similar to the Gronwall Lemma involving sums rather than integrals which may be effectively used in the analysis of finite difference equations. The most precise and complete analogue for the discrete case may be stated as follows.

<u>Lemma 2</u>. Let $\{x_k\}$, $\{f_k\}$, and $\{z_k\}$, $k = 0,1, \ldots, m$, be real valued sequences and let $\{z_k\}$ be nonnegative. If for $k = 0,1, \ldots, m$,

$$x_k \leq f_k + \sum_{0 \leq i < k} z_i x_i, \tag{3}$$

then

$$x_k \leq f_k + \sum_{0 \leq i < k} \Big[\prod_{i < j < k} (1 + z_j) z_i f_i \Big], \tag{4}$$

for $k = 0, 1, \ldots, m$.

Note: In inequality (4) the term $\prod_{i < j < k} A_j (1 + z_j)$ should be interpreted as unity when $k \leq i$. This convention will be adopted throughout this paper. Furthermore, if we are considering a set of matrices $\{A_j\}$, $j = i, \ldots, k$, then $\prod_{i < j < k} A_j$ denotes the product $A_{i+1} A_{i+2} \cdots A_{k-1}$, whereas $\prod_{k > j > i} A_j$ denotes the product $A_{k-1} A_{k-2} \cdots A_{i+1}$.

We observe easily that if c is a constant and $z_i \leqq c$ for $i = 0, 1, \ldots, k-1$, then substituting c for the z_i's in (4) produces the inequality

$$x_k \leqq f_k + c \sum_{0 \leqq i < k} (1 + c)^{k-i-1} f_i.$$

On the other hand, if b is a constant vector such that $f_i \leqq b$ for $i = 0, \ldots, k$, then substituting b for the f_i's in (4) produces the inequality

$$x_k \leqq \prod_{0 \leqq i < k} (1 + z_i) b.$$

Special cases of Lemma 2 are stated and used in the analysis of finite difference equations in [7], [8] and in other readily available literature. Let us now state and prove a lemma which includes Lemma 2 as a special case.

Lemma 3. Let x, f, g, and z be real valued functions defined on an interval $[a, b]$ with g and z nonnegative and let $\tau_o < \tau_1 < \ldots < \tau_m$ be a sequence of numbers in $[a, b]$. If

$$x(t) \leqq f(t) + g(t) \sum_{\tau_i < t} z(\tau_i) x(\tau_i) \tag{5}$$

for all t in $[a, b]$, then for all t in $[a, b]$

$$x(t) \leqq f(t) + g(t) \sum_{\tau_i < t} \Big(\prod_{\tau_i < \tau_j < t} (1 + g(\tau_j) z(\tau_j)) z(\tau_i) f(\tau_i) \Big). \tag{6}$$

Proof. Let y be a function defined on the interval $[a, b]$ by the formula

$$y(t) = f(t) + g(t) \sum_{\tau_i < t} z(\tau_i) y(\tau_i). \tag{7}$$

Obviously the formula

$$y(t) = f(t) + g(t) \sum_{\tau_i < t} \Big(\prod_{\tau_i < \tau_j < \tau_k} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i) \qquad (8)$$

is valid for t in $[a, \tau_1]$ and at all points t where $g(t) = 0$. Let us assume it is satisfied for t in $[a, \tau_k]$. Then for t in $(\tau_k, \min\{\tau_{k+1}, b\}]$ we have

$$\begin{aligned} y(t) = f(t) + g(t)\{ \sum_{\tau_i < \tau_k} z(\tau_i)y(\tau_i) + z(\tau_k)[f(\tau_k) \\ + g(\tau_k) \sum_{\tau_i < \tau_k} \Big(\prod_{\tau_i < \tau_j < \tau_k} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i))]\}. \end{aligned} \qquad (9)$$

If $g(\tau_k) \neq 0$, then our assumption implies

$$\sum_{\tau_i < \tau_k} z(\tau_i)y(\tau_i) = \sum_{\tau_i < \tau_n} \Big(\prod_{\tau_i < \tau_j < \tau_k} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i)),$$

and making the obvious substitution in (9) we observe that formula (8) is satisfied for t in $(\tau_k, \min\{\tau_{k+1}, b\})$. If $g(\tau_i) = 0$ for $k \geq i > k - p$ with either $k - p = 0$ or $g(\tau_{k-p}) \neq 0$, then

$$\begin{aligned} y(t) &= f(t) + g(t)[\sum_{\tau_{k-p} < \tau_i < \tau_{k+1}} \Big(\prod_{\tau_i < \tau_j < \tau_{k+1}} (1+g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i)) \\ &\quad + z(\tau_{k-p})y(\tau_{k-p}) + \sum_{\tau_i < \tau_{k-p}} \Big(\prod_{\tau_i < \tau_j < \tau_{k-p}} (1+g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i))] \\ &= f(t) + g(t) \sum_{\tau_i < \tau} \Big(\prod_{\tau_i < \tau_j < \tau_{k+1}} (1+g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i)). \end{aligned}$$

Hence we may conclude by induction that formula (8) is valid on $[a, b]$.

Now let us define the function $\omega = x - y$. Clearly $\omega(t) \geq 0$ for t in $[a, \tau_1]$. Assume $\omega(t) \geq 0$ on $[a, \tau_k]$ and let t be an arbitrary point in $(\tau_k, \min\{\tau_{k+1}, b\})$. Then

$$\omega(t) = g(t) \sum_{\tau_i \leq \tau_k} z(\tau_i)\omega(\tau_i) \geq 0,$$

and it is immediate from induction that $\omega(t) \geq 0$ on $[a, b]$. This fact together with the previously established validity of formula (8) completes the proof of our lemma.

To illustrate the usefulness of Lemma 3 we will consider a simple application in the theory of finite difference equations. First, however, it is convenient to introduce some additional notations.

Let p be a positive integer and let Ω be a discrete increasing sequence of points in $[-p, \infty)$. For each $t > -p$ let τ_t denote the largest element in Ω less than t and let $\alpha_t = \{\tau : \tau_t \geq \tau \geq \tau_{t-p}, \tau \text{ in } \Omega\}$. We assume that the number of points in α_t is bounded for all $t > -p$. If x is a function defined for $t \geq -p$ with values in R^n (the space of n-dimensional column vectors), then $x(\alpha_t)$ denotes the vector $(x(\nu_1), x(\nu_2), \ldots, x(\nu_k))^T$ in R^{nk} where $\nu_1, \ldots, \nu_k$ is the largest subset of Ω with $\tau_t \geq \nu_k > \nu_{k-1} > \cdots > \nu_1 \geq \tau_{t-p}$. $\| \ \|$ designates any appropriate norm definable on R^m for arbitrary m. If A is an $m \times m$ matrix then $\|A\|$ denotes the smallest number ξ such that $\|Au\| \leq \xi \|u\|$ for all u in R^m.

Let us consider finite difference equations of the form

$$x(t) = \sum_{t-p \leq s \leq t} A(\tau_t, \tau_s)x(\tau_s) + F(\tau_t, x(\alpha_t)), \quad t \geq 0, \qquad (10)$$

where the $A(\tau_t, \tau_s)$'s are $n \times n$ matrices. $F(t, \varphi)$ is a function mapping into R^n and defined for $t \geqq -p$ and all φ in $R^{n(b+1)}$ where b is the maximum number of points in any α_t. Furthermore, we assume there exist a positive constant c and a function $L(t)$ such that

$$\|F(t, \varphi)\| \leq L(t)\|\varphi\| \qquad (11)$$

for all $t \geqq 0$ and all φ such that $\|\varphi\| < c$. It is clear that for each specification of x on $[-p, 0]$ there corresponds a unique solution of equation (10) defined for all $t \geqq 0$.

Now for an arbitrary value of $t \geqq 0$ let $\nu_1 > \nu_2 \cdots > \nu_k$ denote the points in α_t. We define $A^*(\tau_t)$ to be the $(b + 1) \times (b + 1)$ matrix of $n \times n$ matrices $A_{ij}(\tau_t)$ where $A_{1j}(\tau_t) = A(\tau_t, \nu_j)$, $j = 1, \ldots, k$, $A_{i+1,i}(\tau_t) = I$, $i = 2, \ldots, k$, and $A_{ij}(\tau_t) = 0$ otherwise. We define the column vector $x^*(t)$ in $R^{n(b+1)}$ as

$$(x(t), x(\nu_1), \ldots, x(\nu_k), 0, \ldots, 0)^T$$

and the vector $F^*(\tau_t, x^*(\tau_t))$ in $R^{n(b+1)}$ by the formula

$$F^*(\tau_t, x^*(\tau_t)) = (F(\tau_t, x(\alpha_t)), 0, \ldots, 0)^T .$$

It is easily verified that equation (10) is equivalent to the larger system of the form

$$x^*(t) = A^*(\tau_t)x^*(\tau_t) + F^*(\tau_t, x^*(\tau_t)), \quad t \geqq 0. \tag{12}$$

We shall prove the following boundedness result.

<u>Theorem 1</u>. Let $\|\prod_{\substack{\nu \in \Omega \\ t > \nu}} A^*(\nu)\|$ be bounded for all $t \geqq 0$. Suppose there exists a function g such that

$$\sup\{\prod A^*(\nu) : \nu \text{ in } \Omega, \ t > \nu < t_1, \ t \geqq t_1 \geqq 0\} \leqq g(\tau_t) \tag{13}$$

and that $\prod_{\substack{\nu \in \Omega \\ \nu < t}} (1 + g(\nu)L(\nu))$ is bounded for $t \geqq 0$. Then all solutions of equation (10) starting with sufficiently small initial data are bounded. If $c = \infty$ then all solutions of (10) are bounded.

Proof. Let $\nu_0 = 0$ if 0 is contained in Ω and otherwise let $\nu_0 = \tau_0$. Let $\nu_1 < \nu_2 < \ldots$ denote the positive elements of Ω. We observe that

$$x^*(t) = A^*(\nu_0)x^*(\nu_0) + F(\nu_0, x^*(\nu_0))$$

for t in $(0, \nu_1]$, and it follows easily from induction that in general

$$x^*(t) = \prod_{t > \nu_i} A^*(\nu_i)x^*(\nu_0) + \sum_{\nu_i < t} \Big(\prod_{t > \nu_j > \nu_i} A^*(\nu_j)\Big)F(\nu_i, x^*(\nu_i)) \tag{14}$$

for all $t \geqq 0$. Not let u be any real valued function defined on $[-p, \infty)$ such that

$$u(t) \leqq \| \prod_{t > \nu_i} A^*(\nu_i) \| u(\nu_o) + g(\tau_t) \sum_{\nu_i < t} L(\nu_i) u(\nu_i). \tag{15}$$

Employing Lemma 3 we have

$$u(t) \leqq \| \prod_{t < \nu_i} A^*(\nu_i) \| \, u(\nu_o)$$

$$+ g(\tau_t) \sum_{\nu_i < t} \Big(\prod_{\nu_i < \nu_j < t} (1 + g(\nu_j) L(\nu_j)) L(\nu_i) \| \prod_{\nu_i > \nu_j} A^*(\nu_j) \| \, u(\nu_o).$$

By our hypotheses there exists a constant K_1 which bounds $\| \prod_{t > \nu_i} A^*(\nu_i) \|$ for all $t \geqq 0$, so we have

$$u(t) \leqq K_1 \prod_{\nu_i < t} (1 + g(\nu_i) L(\nu_i)) u(\nu_o).$$

But then $\prod_{\nu_i < t} (1 + g(\nu_i) L(\nu_i))$ is by hypothesis also bounded by some constant K_2, so we may conclude that

$$u(t) \leqq K_1 K_2 u(\nu_o) \tag{16}$$

for all $t > 0$. It follows, of course, that $|u(0)| < \frac{c}{K_1 K_2}$ implies $|u(t)| < c$ for all $t \geqq 0$. Now returning to (14) it is clear that (11) and (13) imply that for $\|x^*(\nu_o)\| < \frac{c}{K_1 K_2}$ we have

$$\|x^*(t)\| < \|\prod_{t > \nu_i} A^*(\nu_i)\| \; \|x^*(\nu_o)\| + g(\tau_t) \sum_{\nu_i < t} L(\nu_i)\|x^*(\nu_i)\| \tag{17}$$

which is an inequality of the form of (15). Thus we have by (16) that $\|x^*(t)\| \leq K_1 K_2 \|x^*(\nu_o)\|$ for all $t \geq 0$ and the proof of our Theorem is complete.

A complementary result concerning stability which is essentially proved in the proof of Theorem 1 may be stated as follows.

Theorem 2. Assume the hypotheses of Theorem 1, suppose Ω is not bounded, and suppose $g(t) \to 0$ as $t \to \infty$. Then all solutions of equation (10) starting with sufficiently small initial data tend to zero as $t \to \infty$. If $c = \infty$ then all solutions of (10) tend to zero as $t \to \infty$.

Proof. In the proof of Theorem 1 it was shown that

$$\|x^*(t)\| \leq \|\prod_{t > \nu_i} A^*(\nu_i)\| \; \|x^*(\nu_o)\| + g(\tau_t) \sum_{\nu_i < t} \Big(\prod_{\nu_i < \nu_j < t} (1+g(\nu_j)L(\nu_j)L(\nu_j))L(\nu_i) \tag{18}$$

$$\|\prod_{\nu_i > \nu_j} A^*(\nu_j)\| \; \|x^*(\nu_o)\|,$$

for all sufficiently small initial data and for all initial data if $c = \infty$. Since our hypotheses imply that both terms on the right hand side of inequality (18) tend to zero as $t \to \infty$, Theorem 2 is proved.

For a large number of results related to the boundedness and stability of finite difference equations the reader is referred to [3], [4], and [9]. There is also a wealth of material available in recent Russian literature.

The next step in this development is to establish a lemma which generalizes Lemma 1. However, this generalization is only preparatory for establishing our most all inclusive lemma which contains both Lemma 1 and Lemma 3 as special cases. We shall apply our results in the analysis of a general class of nonlinear integral equations of the Volterra type. Before proceeding, however, let us first review a few of the basic notions involved in constructing Riemann-Stieltjes integrals.

For an interval $[a, b]$ we denote an arbitrary partition of this interval by P. That is, $P = \{t_o, t_1, \ldots, t_m\}$ where $a = t_o < t_1 < \ldots < t_{m-1} < t_m = b$. If f, x, and μ are arbitrary bounded functions defined on the interval $[a, b]$, then a sum of the form

$$S(P, f, \mu) = \sum_{k=1}^{m} f(\tau_k)(\mu(t_k) - \mu(t_{k-1})) \tag{19}$$

is called a Riemann-Stieltjes sum of f with respect to μ. f is referred to as Riemann integrable with respect to μ on $[a, b]$ if there exists a number L having the following properties: For every $\epsilon > 0$, there exists a partition P_ϵ on $[a, b]$ such that for every partition P such that $P_\epsilon \subset P$ and every choice of the points τ_k in $[t_{k-1}, t_k]$, we have $|S(P, f, \mu) - L| < \epsilon$. A n-dimensional vector function $F = (f_1, f_2, \ldots, f_n)$ is referred to as Riemann integrable with respect to μ on $[a, b]$ if such is true for each component function

f_i, $i = 1,2, \ldots, n$. When the limit L (or $(L_1, L_2, \ldots, L_n)$ in the vector case) exists, then it is denoted by $\int_a^b f(\tau)d\mu(\tau)$ (or $\int_a^b F(\tau)d\mu(\tau)$ in the vector case). For an excellent account of the elementary theory of Riemann-Stieltjes integrals the reader is referred to [1].

<u>Lemma 4</u>. Let x, f, g, and z be real valued functions defined on an interval $[a, b]$ and either continuous or of bounded variation. Let g and z be nonnegative and let η be a nondecreasing continuous functional defined on $[a, b]$. If for all t in $[a, b]$

$$x(t) \leq f(t) + g(t) \int_a^t z(\tau)x(\tau)d\eta(\tau), \tag{20}$$

then

$$x(t) \leq f(t) + g(t) \int_a^t f(\tau)z(\tau)\exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right)d\eta(\tau), \tag{21}$$

for all t in $[a, b]$.

Proof. Since the function $\exp\{\int_\tau^t z(s)d\eta(s)\}$ for τ in $[a, t]$ is continuous and of bounded variation, we know from the elementary theory of Riemann-Stieltjes integrals that the function y given by the formula

$$y(t) = f(t) + g(t) \int_a^t f(\tau)z(\tau) \exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right)d\eta(\tau) \tag{22}$$

is well defined and continuous. We will show that

$$-\int_a^t f(\tau)d(\exp(\int_\tau^t z(s)d\eta(s))) = \int_a^t z(\tau)f(\tau)\exp(\int_\tau^t z(s)d\eta(s))d\eta(s). \quad (23)$$

Let $P = (t_o, t_1, \ldots, t_m)$ be an arbitrary partition of $[a, t]$ such that $|t_k - t_{k-1}| < \delta$ for $k = 1,2, \ldots, m$. Considering the corresponding Riemann-Stieltjes sum for the integral on the left hand side of (23) we have

$$\begin{aligned}
&-\sum_{k=1}^m f(\tau_k)[\exp(\int_{t_k}^t z(s)d\eta(s)) - \exp(\int_{t_{k-1}}^t z(s)d\eta(s))] \\
&\quad = \sum_{k=1}^m z(\tau_k)f(\tau_k)\exp(\int_{\tau_k}^t z(s)d\eta(s))(\eta(t_k) - \eta(t_{k-1})) \\
&\qquad + \sum_{k=1}^m z(\tau_k)f(\tau_k)\exp(\int_{\tau_k}^t z(s)d\eta(s)(\exp(-\int_{\tau_k}^{t_k} z(s)d\eta(s)) - 1)(\eta(t_k)- \eta(t_{k-1})) \quad (24) \\
&\qquad + \sum_{k=1}^m f(\tau_k)\exp(\int_{t_k}^t z(s)d\eta(s))(z(\nu_k) - z(\tau_k))(\eta(t_k) - \eta(t_{k-1})) \\
&\qquad + \sum_{k=1}^m f(\tau_k)\exp(\int_{t_k}^t z(s)d\eta(s))(\sum_{i=2}^\infty \frac{1}{i!} z(\nu_k)^i(\eta(t_k) - \eta(t_{k-1}))^i),
\end{aligned}$$

where τ_k and ν_k are points in the interval $[t_{k-1}, t_k]$.
Letting δ take on a sequence of values δ_n, $n = 1, 2, \ldots,$

tending to zero we observe that the only term on the right hand side of the equality sign in (24) which does not tend to zero in the first. But, of course, the first term is just a Riemann-Stieltjes sum for the integral on the right hand side of (23), so we have established the validity of (23).

Let us now show that y as defined by (22) satisfies the equation

$$y(t) = f(t) + g(t) \int_a^t z(\tau)y(\tau)d\eta(\tau) \tag{25}$$

on the interval $[a, b]$. Considering the integral

$$\begin{aligned} I(t) &= \int_a^t z(\tau)\{f(\tau) + g(\tau) \int_a^\tau z(\nu)f(\nu)\exp(\int_\nu^\tau g(s)z(s)d\eta(s))\}d\eta(\tau) \\ &= \int_a^t z(\tau)f(\tau)d\eta(\tau) + \int_a^t z(\tau)g(\tau)\exp(\int_a^\tau g(s)z(s)d\eta(s)) \int_a^\tau z(\nu)f(\nu) \\ &\qquad \exp(-\int_a^\nu g(s)z(s)d\eta(s))d\eta(\tau), \end{aligned}$$

we obtain, using the integrations by parts formula for Riemann-Stieltjes integrals that

$$\begin{aligned} I(t) &= \int_a^t z(\tau)f(\tau)d\eta(\tau) + \int_a^t \{z(\nu)f(\nu)\exp(-\int_a^\nu g(s)z(s)d\eta(s))d\eta(\nu)\} \\ &\qquad d(\exp \int_a^\tau g(s)z(s)d\eta(s)) = y(t) - f(t), \end{aligned}$$

which, of course, implies (25).

Defining $\omega(t) = y(t) - x(t)$ and using (20) and (25) we have

$$\omega(t) \geqq g(t) \int_a^t z(\tau)\omega(\tau)d\eta(\tau). \tag{26}$$

Furthermore, if we define the operator T for bounded functions u on $[a, b]$ by the formula

$$T(u(t)) = g(t) \int_a^t z(\tau)u(\tau)d\eta(\tau),$$

we see that (26) implies

$$\omega(t) \geq T(\omega(t)) \geq T^2(\omega(t)) \geq \ldots \geq T^n(\omega(t)) \geq \ldots, \tag{27}$$

for all t in $[a, b]$ where $T^k(\omega(t)) = T(T^{k-1}(\omega(t)))$. But for arbitrary n,

$$T^n(\omega(t)) \geq \int_a^t z(\tau_1)g(\tau_1) \int_a^{\tau_1} z(\tau_2)g(\tau_2) \ldots \int_a^{\tau_{n-1}} z(\tau_n)\omega(\tau_n)d\eta(\tau_n)d\eta(\tau_{n-1}) \ldots d\eta(\tau_1),$$

so letting $m_1 = \max\{z(t)g(t)\colon\ t \text{ in } [a, b]\}$ and $m_2 = \max\{|\omega(t)| : t \text{ in } [a, b]\}$ we have

$$T^n(\omega(t)) \geq - \frac{m_1^n\, m_2(\eta(t) - \eta(a))^n}{n!}.$$

Hence clearly $T^n(\omega(t)) \to 0$ as $n \to \infty$ and we may conclude from (27) that $\omega(t) \geq 0$ for arbitrary t in $[a, b]$. Therefore, we have proven that $x(t) \leq y(t)$ which was, of course, our objective, and Lemma 3 is established.

Now for the purpose of providing a unified formulation for discrete, continuous, and discontinuous problems, we present a lemma which simultaneously generalizes Lemma 3 and Lemma 4.

<u>Lemma 5</u>. Let x, f, g, and z be real valued functions defined on an interval $[a, b]$ and either continuous or of bounded variation. Let g and z be nonnegative. Let μ be a nondecreasing functional defined on $[a - \epsilon, b]$ for some $\epsilon > 0$ which is continuous from the left, and let x, f, g, and z be continuous from the right at all points of discontinuity of μ. If for all t in $[a, b]$

$$x(t) \leq f(t) + g(t) \int_a^{t-} z(\tau)x(\tau)d\mu(\tau), \tag{27}$$

then

$$x(t) \leq f(t) + g(t) \int_a^{t-} f(\tau)z(\tau)\exp\left(\int_\tau^{t-} g(s)z(s)d\mu(s)\right)d\mu(\tau), \tag{28}$$

for all t in $[a, b]$. Let $\zeta_o = a$, let $\{\zeta_i\}$, $i = 1,2, \ldots$ denote the set of discontinuities of μ in $(a, b]$, and for all t in $[a, b]$ let

$$\eta(t) = \mu(t) - \sum_{\zeta_i < t} J(\zeta_i), \tag{29}$$

when $J(t) = \mu(t+) - \mu(t)$. Then for all t in $[a, b]$,

$$x(t) \leq f(t) + g(t) \int_a^{t-} \Big(\prod_{\tau < \zeta_i < t} (1+g(\zeta_i)z(\zeta_i)J(\zeta_i))\exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right) z(\tau)f(\tau)d\mu(\tau). \tag{30}$$

Furthermore if $|f(t)|$ and $g(t)$ are bounded on $[a, b]$ by constants K and c respectively, then

$$x(t) \leq K \prod_{a < \zeta_i < t} (1 + cz(\zeta_i)J(\zeta_i))\exp\left(c \int_a^t z(s)d\eta(s)\right) \tag{31}$$

for all t in $[a, b]$.

Proof. Clearly we have by hypothesis that

$$x(t) \leq f(t) + g(t) \int_a^t z(\tau)x(\tau)d\eta(\tau) + g(t) \sum_{\zeta_i < t} z(\zeta_i)x(\zeta_i)J(\zeta_i).$$

Since η is obviously continuous as well as nondecreasing it follows from Lemma 4 that

$$x(t) \leq f(t) + g(t) \sum_{\zeta_i < t} z(\zeta_i)x(\zeta_i)J(\zeta_i)$$
$$+ g(t) \int_a^t z(\tau)\exp(\int_\tau^t g(s)z(s)d\eta(s))(f(\tau) + g(\tau) \sum_{\zeta_i < \tau} z(\zeta_j)x(\zeta_j)J(\zeta_j))d\eta(\tau),$$

and consequently

$$x(t) \leq f(t) + g(t) \sum_{\zeta_i < t} z(\zeta_j)x(\zeta_j)J(\zeta_j) + g(t) \int_a^t z(\tau)\exp(\int_\tau^t g(s)z(s)d\eta(s))f(\tau)d\eta(\tau)$$
$$+ g(t)\exp(\int_a^t g(\tau)z(\tau)d\eta(\tau))I(t), \tag{32}$$

where

$$I(t) = \int_a^t z(\tau)g(\tau)\exp(- \int_a^\tau g(s)z(s)d\eta(s)) \sum_{\zeta_i < \tau} z(\zeta_i)x(\zeta_i)J(\zeta_i)d\eta(\tau).$$

Using the integration by parts formula we have

$$I(t) = - \exp(- \int_a^t g(\tau)z(\tau)d\eta(\tau)) \sum_{\zeta_i < t} z(\zeta_i)x(\zeta_i)J(\zeta_i)$$
$$+ \sum_{\zeta_i < t} \exp(- \int_a^{\zeta_i} g(s)z(s)d\eta(s))z(\zeta_i)x(\zeta_i)J(\zeta_i),$$

and substituting in (32) it follows that

$$x(t) \leq f(t) + g(t) \int_a^t z(\tau)\exp(\int_\tau^t g(s)z(s)d\eta(s))f(\tau)d\eta(\tau)$$
$$+ g(t) \sum_{\zeta_i < t} \exp(\int_{\zeta_i}^t g(\tau)z(\tau)d\eta(\tau))z(\zeta_i)J(\zeta_i)x(\zeta_i). \tag{33}$$

Now for arbitrary $\delta > 0$ we can choose a finite subset $\{\nu_i\}$, $i = 0,1, \ldots, m$, of $\{\zeta_i\}$, $i = 0,1, \ldots,$ such that $\nu_o = a$ and

$$g(t) \sum_{\zeta_i < t} \exp\left(\int_{\zeta_i}^{t} g(s)z(s)d\eta(s)\right)z(\zeta_i)J(\zeta_i)x(\zeta_i)$$
$$= g(t) \sum_{\nu_i < t} \exp\left(\int_{\nu_i}^{t} g(s)z(s)d\eta(s)\right)z(\nu_i)J(\nu_i)x(\nu_i) + \delta(t) \tag{34}$$

where $|\delta(t)| < \delta$ for all t in $[a, b]$. Substituting (34) into (33) and envoking Lemma 3 it follows that

$$x(t) \leq F_\delta(t) + g(t) \sum_{\nu_i < t} \left(\prod_{\nu_i < \nu_j < t} (1+g(\nu_j)z(\nu_j)J(\nu_j)) \right.$$
$$\left. \exp\left(\int_{\nu_i}^{t} g(s)z(s)d\eta(s)\right)z(\nu_i)J(\nu_i)F_\delta(\tau_j) \right), \tag{35}$$

where

$$F_\delta(t) = f(t) + g(t) \int_a^t z(\tau)\exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right)f(\tau)d\eta(\tau) + \delta(t).$$

Again using the integration by parts formula we may observe that

$$g(t) \sum_{\nu_i < t} \left(\prod_{\nu_i < \nu_j < t} (1+g(\nu_j)z(\nu_j)J(\nu_j))g(\nu_i)z(\nu_i)J(\nu_i) \right.$$
$$\left. \int_a^{\nu_i} \exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right)z(\tau)f(\tau)d\eta(\tau)\right) \tag{36}$$
$$= - g(t) \int_a^{\tau_t} \exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right)z(\tau)f(\tau)d\eta(\tau)$$
$$+ g(t) \sum_{\nu_i < \tau_t} \left(\prod_{\nu_i < \nu_j < t} (1+g(\nu_j)z(\nu_j)J(\nu_j))\left(\int_{\nu_i}^{\nu_{i+1}} \exp\left(\int_\tau^t g(s)z(s)d\eta(s)\right)\right.\right.$$
$$\left.\left. z(\tau)f(\tau)d\eta(\tau)\right)\right),$$

where τ_t denotes the largest member of $\{\nu_i\}$, $i = 0,1, \ldots, m$ which is less than t. Now substituting (36) into (35) we have that

$$x(t) \leq \delta_1(t) + f(t) + g(t) \int_{\tau_t}^{t} \exp(\int_{\tau}^{t} g(s)z(s)d\eta(s))z(\tau)f(\tau)d\eta(\tau)$$

$$+ g(t) \sum_{\nu_i < t} (\prod_{\nu_i < \nu_j < t} (1+g(\nu_j)z(\nu_j)J(\nu_j))\exp(\int_{\nu_i}^{t} g(s)z(s)d\eta(s))z(\nu_i)f(\nu_i)J(\nu_i)) \tag{37}$$

$$+ g(t) \sum_{\nu_i < \tau_t} \int_{\nu_i}^{\nu_{i+1}} (\prod_{\nu_i < \nu_j < t} (1+g(\nu_j)z(\nu_j)J(\nu_j))\exp(\int_{\tau}^{t} g(s)z(s)d\eta(s))z(\tau)f(\tau)d\eta(\tau)),$$

where

$$\delta_1(t) = \delta(t) + g(t) \sum_{\nu_i < t} (\prod_{\nu_i < \nu_j < t} (1 + g(\nu_j)z(\nu_j)J(\nu_j))$$

$$\exp(\int_{\nu_i}^{t} g(s)z(s)d\eta(s))z(\nu_i)J(\nu_i)\delta(\nu_i).$$

Furthermore, it is easily verified that the right hand side of (37) reduces to yield the inequality

$$x(t) \leq f(t)+g(t) \int_{a}^{t-} (\prod_{\tau < \nu_i < t} (1 + g(\nu_i)z(\nu_i)J(\nu_i))$$

$$\exp(\int_{\tau}^{t} g(s)z(s)d\eta(s))z(\tau)f(\tau)d\mu(\tau) + \delta_1(t).$$

Since $\delta_1(t)$ is bounded on $[a, b]$ by a constant times δ and δ may be chosen arbitrarily small, it follows that

$$x(t) \le f(t) + g(t) \int_a^{t-} \Big(\prod_{\tau < \zeta_i < t} (1 + g(\zeta_i)z(\zeta_i)J(\zeta_i)) \qquad (38)$$
$$\exp\Big(\int_\tau^t g(s)z(s)d\eta(s)\Big)z(\tau)f(\tau)d\mu(\tau).$$

Making the observation that

$$\prod_{\tau < \zeta_i < t} (1 + g(\zeta_i)z(\zeta_i)J(\zeta_i)) \le \exp\Big(\sum_{\tau < \zeta_i < t} g(\zeta_i)z(\zeta_i)J(\zeta_i)\Big),$$

we see that (38) implies

$$x(t) \le f(t) + g(t) \int_a^{t-} \exp\Big(\int_\tau^{t-} g(s)z(s)d\mu(s)\Big)z(\tau)f(\tau)d\mu(\tau).$$

Substituting the bounds K and c for $|f(t)|$ and $g(t)$ respectively in (38), we may verify that

$$x(t) \le K \prod_{a < \zeta_i < t} (1 + cz(\zeta_i)J(\zeta_i))\exp\Big(c \int_a^t z(s)d\eta(s)\Big),$$

and the proof of our lemma is complete.

One may verify that Lemma 5 similar to our previous lemmas presents a "best possible" inequality. That is, if the inequality sign in (27) is replaced by equality then the same may be done in (30).

Consider now a function $K(t, \tau, \varphi)$ mapping $[0, \infty) \times [0, \infty) \times R^n$ into R^n which is either continuous in τ or of bounded variation on bounded intervals in τ. Suppose there exists positive functions g and L on $[0, \infty)$ which are either continuous or of bounded variation on bounded intervals and are such that for all φ_1 and φ_2 in R^n,

$$\|K(t, \tau, \varphi_1) - K(t, \tau, \varphi_2)\| \leq g(t)L(\tau)\|\varphi_1 - \varphi_2\|. \tag{39}$$

As a straightforward application of Lemma 5 we shall present a result concerning Volterra integral equations of the form

$$x(t) = f(t) + \int_0^{t-} K(t, \tau, x(\tau))d\mu(\tau), \qquad t \geq 0, \tag{40}$$

when f is either continuous or of bounded variation on bounded intervals and μ is nondecreasing and continuous from the left. We assume g, f, and L are continuous from the right at all points of discontinuity of μ.

Theorem 3. Suppose equation (40) has a bounded solution x defined on $[0, \infty)$. Suppose g is bounded and

$$\int_0^{\infty} L(\tau)d\eta(\tau) < \infty. \tag{41}$$

If h is a function defined on $[0, \infty)$ which is either continuous or of bounded variations on bounded intervals and such that $\|f(t) - h(t)\|$ is bounded on $[0, \infty)$, then any solution of the equation

$$y(t) = h(t) + \int_0^{t-} K(t, \tau, y(\tau))d\mu(\tau) \tag{42}$$

is bounded. Furthermore, if $g(t) \to 0$ and $\|f(t) - h(t)\| \to 0$ as $t \to \infty$, then $\|x(t) - y(t)\| \to 0$ as $t \to \infty$.

Proof. Clearly

$$x(t) - y(t) = f(t) - h(t) + \int_0^{t-} (K(t, \tau, x(\tau)) - K(t, \tau, y(\tau))d\mu(\tau),$$

so by (39) we have

$$\|x(t) - y(t)\| \leq \|f(t) - h(t)\| + g(t) \int_0^{t-} L(\tau)\|x(\tau) - y(\tau)\|d\mu(\tau). \tag{43}$$

Hence envoking Lemma 5 we have

$$\begin{aligned}\|x(t) - y(t)\| &\leq \|f(t) - h(t)\| \\ &+ g(t) \int_a^{t-} \Big(\prod_{\tau < \zeta_i < t} (1+g(\zeta_i)L(\zeta_i)J(\zeta_i))\exp(\int_\tau^t g(s)L(s)d\eta(s)L(\tau)\|f(\tau)-h(\tau)\|d\mu(\tau)\end{aligned} \tag{44}$$

where $\{\zeta_i\}$, $i = 1,2, \ldots$ denote the discontinuities in μ,

$$\eta(t) = \mu(t) - \sum_{\zeta_i < t} J(\zeta_i),$$

and $J(t) = \mu(t+) - \mu(t)$. By hypothesis there exist constants b and c which bound $g(t)$ and $\|f(t) - h(t)\|$ on $[0, \infty)$ respectively, so we conclude from (44) that

$$\begin{aligned}\|x(t) - y(t)\| &\leq c \prod_{\zeta_i < t} (1 + bL(\zeta_i)J(\zeta_i))\exp(b \int_a^t L(s)d\eta(s)) \\ &\leq c \exp (b \int_0^{t-} L(\tau)d\mu(\tau)).\end{aligned} \tag{ 5}$$

Since $\int_0^\infty L(\tau)d\eta(\tau) < \infty$ and x is bounded on $[0, \infty)$, it follows that y is bounded on $[0, \infty)$. We also observe from (44) that

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$$\|x(t) - y(t)\| \leqq \|f(t) - h(t)\| + \frac{cg(t)}{b} (\exp(b \int_0^{t-} L(\tau)d\mu(\tau)) - 1).$$

Thus it is clear that if $g(t) \to 0$ and $\|f(t) - h(t)\| \to 0$ as $t \to \infty$ then $\|x(t) - y(t)\| \to 0$ as $t \to \infty$ and our theorem is proved.